CHAPTER 14

Integral Calculus: The Indefinite Integral

14.1 INTEGRATION

Chapters 3 to 6 were devoted to differential calculus, which measures the rate of change of functions. Differentiation, we learned, is the process of finding the derivative $F'(x)$ of a function $F(x)$. Frequently in economics, however, we know the rate of change of a function $F'(x)$ and want to find the original function. Reversing the process of differentiation and finding the original function from the derivative is called *integration*, or *antidifferentiation*. The original function $F(x)$ is called the *integral*, or *antiderivative*, of $F'(x)$.

EXAMPLE 1. Letting $f(x) = F'(x)$ for simplicity, the antiderivative of $f(x)$ is expressed mathematically as

$$\int f(x)\,dx = F(x) + c$$

Here the left-hand side of the equation is read, "the *indefinite integral* of f of x with respect to x." The symbol $\int$ is an *integral sign*, $f(x)$ is the *integrand*, and c is the *constant of integration*, which is explained in Example 3.

14.2 RULES OF INTEGRATION

The following rules of integration are obtained by reversing the corresponding rules of differentiation. Their accuracy is easily checked, since the derivative of the integral must equal the integrand. Each rule is illustrated in Example 2 and Problems 14.1 to 14.6.

Rule 1. The integral of a constant k is

$$\int k\,dx = kx + c$$

Rule 2. The integral of 1, written simply as dx, not $1\,dx$, is

$$\int dx = x + c$$

Rule 3. The integral of a power function x^n, where $n \neq -1$, is given by the *power rule*:

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + c \qquad n \neq -1$$

Rule 4. The integral of x^{-1} (or $1/x$) is

$$\int x^{-1}\,dx = \ln x + c \qquad x > 0$$

The condition $x > 0$ is added because only positive numbers have logarithms. For negative numbers,

$$\int x^{-1}\,dx = \ln|x| + c \qquad x \neq 0$$

Rule 5. The integral of an exponential function is

$$\int a^{kx}\,dx = \frac{a^{kx}}{k\ln a} + c$$

Rule 6. The integral of a natural exponential function is

$$\int e^{kx}\,dx = \frac{e^{kx}}{k} + c \qquad \text{since} \qquad \ln e = 1$$

Rule 7. The integral of a constant times a function equals the constant times the integral of the function.

$$\int kf(x)\,dx = k\int f(x)\,dx$$

Rule 8. The integral of the sum or difference of two or more functions equals the sum or difference of their integrals.

$$\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$$

Rule 9. The integral of the negative of a function equals the negative of the integral of that function.

$$\int -f(x)\,dx = -\int f(x)\,dx$$

EXAMPLE 2. The rules of integration are illustrated below. Check each answer on your own by making sure that the derivative of the integral equals the integrand.

i) $$\int 3\,dx = 3x + c \qquad \text{(Rule 1)}$$

ii) $$\int x^2\,dx = \frac{1}{2+1}x^{2+1} + c = \frac{1}{3}x^3 + c \qquad \text{(Rule 3)}$$

iii)
$$\int 5x^4\,dx = 5\int x^4\,dx \qquad \text{(Rule 7)}$$
$$= 5\left(\frac{1}{5}x^5 + c_1\right) \qquad \text{(Rule 3)}$$
$$= x^5 + c$$

where c_1 and c are arbitrary constants and $5c_1 = c$. Since c is an arbitrary constant, it can be ignored in the preliminary calculation and included only in the final solution.

iv)
$$\int (3x^3 - x + 1)\,dx = 3\int x^3\,dx - \int x\,dx + \int dx \qquad \text{(Rules 7, 8, and 9)}$$
$$= 3(\tfrac{1}{4}x^4) - \tfrac{1}{2}x^2 + x + c \qquad \text{(Rules 2 and 3)}$$
$$= \tfrac{3}{4}x^4 - \tfrac{1}{2}x^2 + x + c$$

v)
$$\int 3x^{-1}\,dx = 3\int x^{-1}\,dx \qquad \text{(Rule 7)}$$
$$= 3\ln|x| + c \qquad \text{(Rule 4)}$$

vi)
$$\int 2^{3x}\,dx = \frac{2^{3x}}{3\ln 2} + c \qquad \text{(Rule 5)}$$

vii)
$$\int 9e^{-3x}\,dx = \frac{9e^{-3x}}{-3} + c \qquad \text{(Rule 6)}$$
$$= -3e^{-3x} + c$$

EXAMPLE 3. Functions which differ by only a constant have the same derivative. The function $F(x) = 2x + k$ has the same derivative, $F'(x) = f(x) = 2$, for any infinite number of possible values for k. If the process is reversed, it is clear that $\int 2\,dx$ must be the antiderivative or indefinite integral for an infinite number of functions differing from each other by only a constant. The constant of integration c thus represents the value of any constant which was part of the primitive function but precluded from the derivative by the rules of differentiation.

The graph of an indefinite integral $\int f(x)\,dx = F(x) + c$, where c is unspecified, is a family of curves parallel in the sense that the slope of the tangent to any of them at x is $f(x)$. Specifying c specifies the curve; changing c shifts the curve. This is illustrated in Fig. 14-1 for the indefinite integral $\int 2\,dx = 2x + c$ where $c = -7, -3, 1$, and 5, respectively. If $c = 0$, the curve begins at the origin.

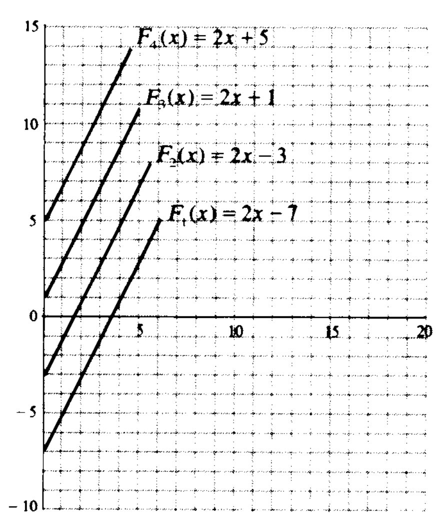

Fig. 14-1

14.3 INITIAL CONDITIONS AND BOUNDARY CONDITIONS

In many problems an *initial condition* ($y = y_0$ when $x = 0$) or a *boundary condition* ($y = y_0$ when $x = x_0$) is given which uniquely determines the constant of integration. By permitting a unique determination of c, the initial or boundary condition singles out a specific curve from the family of curves illustrated in Example 3 and Problems 14.3 to 14.5.

EXAMPLE 4. Given the boundary condition $y = 11$ when $x = 3$, the integral $y = \int 2\,dx$ is evaluated as follows:

$$y = \int 2\,dx = 2x + c$$

Substituting $y = 11$ when $x = 3$,

$$11 = 2(3) + c \qquad c = 5$$

Therefore, $y = 2x + 5$. Note that even though c is specified, $\int 2\,dx$ remains an *indefinite* integral because x is unspecified. Thus, the integral $2x + 5$ can assume an infinite number of possible values.

14.4 INTEGRATION BY SUBSTITUTION

Integration of a product or quotient of two differentiable functions of x, such as

$$\int 12x^2(x^3 + 2)\,dx$$

cannot be done directly by using the simple rules above. However, if the integrand can be expressed as a *constant* multiple of another function u and its derivative du/dx, integration by substitution is possible. By expressing the integrand $f(x)$ as a function of u and its derivative du/dx and integrating with respect to x,

$$\int f(x)\,dx = \int \left(u\frac{du}{dx}\right) dx$$

$$\int f(x)\,dx = \int u\,du = F(u) + c$$

The substitution method reverses the operation of the chain rule and the generalized power function rule in differential calculus. See Examples 5 and 6 and Problems 14.7 to 14.18.

EXAMPLE 5. The substitution method is used below to determine the indefinite integral

$$\int 12x^2(x^3 + 2)\,dx$$

1. Be sure that the integrand can be converted to a product of another function u and its derivative du/dx times a *constant* multiple. (a) Let u equal the function in which the independent variable is raised to the higher power in terms of absolute value; here let $u = x^3 + 2$. (b) Take the derivative of u; $du/dx = 3x^2$. (c) Solve algebraically for dx; $dx = du/3x^2$. (d) Then substitute u for $x^3 + 2$ and $du/3x^2$ for dx in the original integrand:

$$\int 12x^2(x^3 + 2)\,dx = \int 12x^2 \cdot u \cdot \frac{du}{3x^2} = \int 4u\,du = 4\int u\,du$$

   where 4 is a *constant* multiple of u.

2. Integrate with respect to u, using Rule 3 and ignoring c in the first step of the calculation.

$$4\int u\,du = 4(\tfrac{1}{2}u^2) = 2u^2 + c$$

3. Convert back to the terms of the original problem by substituting x^3+2 for u.

$$\int 12x^2(x^3+2)\,dx = 2u^2 + c = 2(x^3+2)^2 + c$$

4. Check the answer by differentiating with the generalized power function rule or chain rule.

$$\frac{d}{dx}[2(x^3+2)^2 + c] = 4(x^3+2)(3x^2) = 12x^2(x^3+2)$$

See also Problems 14.7 to 14.18.

EXAMPLE 6. Determine the integral $\int 4x(x+1)^3\,dx$.

Let $u = x+1$. Then $du/dx = 1$ and $dx = du/1 = du$. Substitute $u = x+1$ and $dx = du$ in the original integrand.

$$\int 4x(x+1)^3\,dx = \int 4xu^3\,du = 4\int xu^3\,du$$

Since x is a *variable* multiple which cannot be factored out, the original integrand cannot be transformed to a *constant* multiple of $u\,du/dx$. Hence the substitution method is ineffectual. Integration by parts (Section 14.5) may be helpful.

14.5 INTEGRATION BY PARTS

If an integrand is a product or quotient of differentiable functions of x and cannot be expressed as a constant multiple of $u\,du/dx$, integration by parts is frequently useful. The method is derived by reversing the process of differentiating a product. From the product rule in Section 3.7.5,

$$\frac{d}{dx}[f(x)\,g(x)] = f(x)\,g'(x) + g(x)\,f'(x)$$

Taking the integral of the derivative gives

$$f(x)\,g(x) = \int f(x)\,g'(x)\,dx + \int g(x)\,f'(x)\,dx$$

Then solving algebraically for the first integral on the right-hand side,

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int g(x)f'(x)\,dx \qquad (14.1)$$

See Examples 7 and 8 and Problems 14.19 to 14.24.

For more complicated functions, *integration tables* are generally used. Integration tables provide formulas for the integrals of as many as 500 different functions, and they can be found in mathematical handbooks.

EXAMPLE 7. Integration by parts is used below to determine

$$\int 4x(x+1)^3\,dx$$

1. Separate the integrand into two parts amenable to the formula in (*14.1*). As a general rule, consider first the simpler function for $f(x)$ and the more complicated function for $g'(x)$. By letting $f(x) = 4x$ and $g'(x) = (x+1)^3$, then $f'(x) = 4$ and $g(x) = \int (x+1)^3\,dx$, which can be integrated by using the simple power rule (Rule 3):

$$g(x) = \int (x+1)^3\,dx = \tfrac{1}{4}(x+1)^4 + c_1$$

2. Substitute the values for $f(x)$, $f'(x)$, and $g(x)$ in (*14.1*); and note that $g'(x)$ is not used in the formula.

$$\int 4x(x+1)^3\,dx = f(x)\cdot g(x) - \int [g(x)\cdot f'(x)]\,dx$$
$$= 4x[\tfrac{1}{4}(x+1)^4 + c_1] - \int [\tfrac{1}{4}(x+1)^4 + c_1](4)\,dx$$
$$= x(x+1)^4 + 4c_1 x - \int [(x+1)^4 + 4c_1]\,dx$$

3. Use Rule 3 to compute the final integral and substitute.

$$\int 4x(x+1)^3\,dx = x(x+1)^4 + 4c_1 x - \tfrac{1}{5}(x+1)^5 - 4c_1 x + c$$
$$= x(x+1)^4 - \tfrac{1}{5}(x+1)^5 + c$$

Note that the c_1 term does not appear in the final solution. Since this is common to integration by parts, c_1 will henceforth be assumed equal to 0 and not formally included in future problem solving.

4. Check the answer by letting $y(x) = x(x+1)^4 - \frac{1}{5}(x+1)^5 + c$ and using the product and generalized power function rules.

$$y'(x) = [x\cdot 4(x+1)^3 + (x+1)^4\cdot 1] - (x+1)^4 = 4x(x+1)^3$$

EXAMPLE 8. The integral $\int 2xe^x\,dx$ is determined as follows:
Let $f(x) = 2x$ and $g'(x) = e^x$; then $f'(x) = 2$, and by Rule 6, $g(x) = \int e^x\,dx = e^x$. Substitute in (*14.1*).

$$\int 2xe^x\,dx = f(x)\cdot g(x) - \int g(x)\cdot f'(x)\,dx$$
$$= 2x\cdot e^x - \int e^x\cdot 2\,dx = 2xe^x - 2\int e^x\,dx$$

Apply Rule 6 again and remember the constant of integration.

$$\int 2xe^x\,dx = 2xe^x - 2e^x + c$$

Then let $y(x) = 2xe^x - 2e^x + c$ and check the answer.

$$y'(x) = 2x\cdot e^x + e^x\cdot 2 - 2e^x = 2xe^x$$

14.6 ECONOMIC APPLICATIONS

Net investment I is defined as the rate of change in capital stock formation K over time t. If the process of capital formation is continuous over time, $I(t) = dK(t)/dt = K'(t)$. From the rate of investment, the level of capital stock can be estimated. Capital stock is the integral with respect to time of net investment:

$$K_t = \int I(t)\,dt = K(t) + c = K(t) + K_0$$

where c = the initial capital stock K_0.

Similarly, the integral can be used to estimate total cost from marginal cost. Since marginal cost is the change in total cost from an incremental change in output, MC = dTC/dQ, and only variable costs change with the level of output

$$\text{TC} = \int \text{MC}\,dQ = \text{VC} + c = \text{VC} + \text{FC}$$

since c = the fixed or initial cost FC. Economic analysis which traces the time path of variables or attempts to determine whether variables will converge toward equilibrium over time is called *dynamics*. For similar applications, see Example 9 and Problems 14.25 to 14.35.

EXAMPLE 9. The rate of net investment is given by $I(t) = 140t^{3/4}$, and the initial stock of capital at $t = 0$ is 150. Determining the function for capital K, the time path $K(t)$,

$$K = \int 140t^{3/4}\,dt = 140\int t^{3/4}\,dt$$

By the power rule,

$$K = 140(\tfrac{4}{7}t^{7/4}) + c = 80t^{7/4} + c$$

But $c = K_0 = 150$. Therefore, $K = 80t^{7/4} + 150$.

Solved Problems

INDEFINITE INTEGRALS

14.1. Determine the following integrals. Check the answers on your own by making sure that the derivative of the integral equals the integrand.

a) $\int 3.5\,dx$

$$\int 3.5\,dx = 3.5x + c \qquad \text{(Rule 1)}$$

b) $\int -\frac{1}{2}\,dx$

$$\int -\tfrac{1}{2}\,dx = -\int \tfrac{1}{2}\,dx = -\tfrac{1}{2}x + c \qquad \text{(Rules 1 and 9)}$$

c) $\int dx$

$$\int dx = x + c \qquad \text{(Rule 2)}$$

d) $\int x^5\,dx$

$$\int x^5\,dx = \tfrac{1}{6}x^6 + c \qquad \text{(Rule 3)}$$

e) $\int 4x^3\,dx$

$$\int 4x^3\,dx = 4\int x^3\,dx \qquad \text{(Rule 7)}$$

$$= 4(\tfrac{1}{4}x^4) + c = x^4 + c \qquad \text{(Rule 3)}$$

f) $\int x^{2/3}\,dx$

$$\int x^{2/3}\,dx = \tfrac{3}{5}x^{5/3} + c \qquad \text{(Rule 3)}$$

g) $\int x^{-1/5}\,dx$

$$\int x^{-1/5}\,dx = \tfrac{5}{4}x^{4/5} + c \qquad \text{(Rule 3)}$$

h) $\int 4x^{-2}\,dx$

$$\int 4x^{-2}\,dx = -4x^{-1} + c = -\frac{4}{x} + c \qquad \text{(Rule 3)}$$

i) $\int x^{-5/2}\,dx$

$$\int x^{-5/2}\,dx = -\frac{2}{3}x^{-3/2} + c = \frac{-2}{3\sqrt{x^3}} + c \qquad \text{(Rule 3)}$$

14.2. Redo Problem 14.1 for each of the following:

a) $\int \frac{dx}{x}$

$$\int \frac{dx}{x} = \int \frac{1}{x}\,dx = \ln|x| + c \qquad \text{(Rule 4)}$$

b) $\int 5x^{-1}\,dx$

$$\int 5x^{-1}\,dx = 5\ln|x| + c \qquad \text{(Rules 7 and 4)}$$

c) $\int \frac{1}{3x}\,dx$

$$\int \frac{1}{3x}\,dx = \frac{1}{3}\int \frac{1}{x}\,dx = \frac{1}{3}\ln|x| + c \qquad \text{(Rules 7 and 4)}$$

d) $\int \sqrt{x}\,dx$

$$\int \sqrt{x}\,dx = \int x^{1/2}\,dx = \tfrac{2}{3}x^{3/2} + c \qquad \text{(Rule 3)}$$

e) $\int \frac{dx}{x^4}$

$$\int \frac{dx}{x^4} = \int x^{-4}\,dx = -\frac{1}{3}x^{-3} + c \qquad \text{(Rule 3)}$$

f) $\int \frac{dx}{\sqrt[3]{x}}$

$$\int \frac{dx}{\sqrt[3]{x}} = \int x^{-1/3}\,dx = \frac{3}{2}x^{2/3} + c \qquad \text{(Rule 3)}$$

g) $\int (5x^3 + 2x^2 + 3x)\,dx$

$$\int (5x^3 + 2x^2 + 3x)\,dx = 5\int x^3\,dx + 2\int x^2\,dx + 3\int x\,dx \qquad \text{(Rules 7 and 8)}$$

$$= 5(\tfrac{1}{4}x^4) + 2(\tfrac{1}{3}x^3) + 3(\tfrac{1}{2}x^2) + c \qquad \text{(Rule 3)}$$

$$= \tfrac{5}{4}x^4 + \tfrac{2}{3}x^3 + \tfrac{3}{2}x^2 + c$$

h) $\int (2x^6 - 3x^4)\,dx$

$$\int (2x^6 - 3x^4)\,dx = \tfrac{2}{7}x^7 - \tfrac{3}{5}x^5 + c \qquad \text{(Rules 3, 7, 8, and 9)}$$

14.3. Find the integral for $y = \int (x^{1/2} + 3x^{-1/2})\,dx$, given the initial condition $y = 0$ when $x = 0$.

$$y = \int (x^{1/2} + 3x^{-1/2})\,dx = \tfrac{2}{3}x^{3/2} + 6x^{1/2} + c$$

Substituting the initial condition $y = 0$ when $x = 0$ above, $c = 0$. Hence, $y = \tfrac{2}{3}x^{3/2} + 6x^{1/2}$.

14.4. Find the integral for $y = \int (2x^5 - 3x^{-1/4})\,dx$, given the initial condition $y = 6$ when $x = 0$.

$$y = \int (2x^5 - 3x^{-1/4})\,dx = \tfrac{1}{3}x^6 - 4x^{3/4} + c$$

Substituting $y = 6$ and $x = 0$, $c = 6$. Thus, $y = \tfrac{1}{3}x^6 - 4x^{3/4} + 6$.

14.5. Find the integral for $y = \int (10x^4 - 3)\,dx$, given the boundary condition $y = 21$ when $x = 1$.

$$y = \int (10x^4 - 3)\,dx = 2x^5 - 3x + c$$

Substituting $y = 21$ and $x = 1$, $21 = 2(1)^5 - 3(1) + c \qquad c = 22$

$$y = 2x^5 - 3x + 22$$

14.6. Redo Problem 14.1 for each of the following:

a) $\int 2^{4x}\,dx$

$$\int 2^{4x}\,dx = \frac{2^{4x}}{4\ln 2} + c \qquad \text{(Rule 5)}$$

b) $\int 8^x\,dx$

$$\int 8^x\,dx = \frac{8^x}{\ln 8} + c$$

c) $\int e^{5x}\,dx$

$$\int e^{5x}\,dx = \frac{e^{5x}}{5} + c \qquad \text{(Rule 6)}$$
$$= \tfrac{1}{5}e^{5x} + c$$

d) $\int 16e^{-4x}\,dx$

$$\int 16e^{-4x}\,dx = \frac{16e^{-4x}}{-4} + c = -4e^{-4x} + c$$

e) $\int (6e^{3x} - 8e^{-2x})\,dx$

$$\int (6e^{3x} - 8e^{-2x})\,dx = \frac{6e^{3x}}{3} - \frac{8e^{-2x}}{-2} + c = 2e^{3x} + 4e^{-2x} + c$$

INTEGRATION BY SUBSTITUTION

14.7. Determine the following integral, using the substitution method. Check the answer on your own. Given $\int 10x(x^2 + 3)^4\,dx$.

Let $u = x^2 + 3$. Then $du/dx = 2x$ and $dx = du/2x$. Substituting in the original integrand to reduce it to a function of $u\,du/dx$,

$$\int 10x(x^2+3)^4\,dx = \int 10xu^4\frac{du}{2x} = 5\int u^4\,du$$

Integrating by the power rule, $\quad 5\int u^4\,du = 5(\tfrac{1}{5}u^5) = u^5 + c$

Substituting $u = x^2 + 3$, $\quad \int 10x(x^2+3)^4\,dx = u^5 + c = (x^2+3)^5 + c$

14.8. Redo Problem 14.7, given $\int x^4(2x^5-5)^4\,dx$.

Let $u = 2x^5 - 5$, $du/dx = 10x^4$, and $dx = du/10x^4$. Substituting in the original integrand,

$$\int x^4(2x^5-5)^4\,dx = \int x^4u^4\frac{du}{10x^4} = \frac{1}{10}\int u^4\,du$$

Integrating, $\quad \dfrac{1}{10}\displaystyle\int u^4\,du = \frac{1}{10}\left(\frac{1}{5}u^5\right) = \frac{1}{50}u^5 + c$

Substituting, $\quad \displaystyle\int x^4(2x^5-5)^4\,dx = \frac{1}{50}u^5 + c = \frac{1}{50}(2x^5-5)^5 + c$

14.9. Redo Problem 14.7, given $\int (x-9)^{7/4}\,dx$.

Let $u = x - 9$. Then $du/dx = 1$ and $dx = du$. Substituting,

$$\int (x-9)^{7/4}\,dx = \int u^{7/4}\,du$$

Integrating, $\quad \displaystyle\int u^{7/4}\,du = \frac{4}{11}u^{11/4} + c$

Substituting, $\quad \displaystyle\int (x-9)^{7/4}\,dx = \frac{4}{11}(x-9)^{11/4} + c$

Whenever $du/dx = 1$, the power rule can be used immediately for integration by substitution.

14.10. Redo Problem 14.7, given $\int (6x-11)^{-5}\,dx$.

Let $u = 6x - 11$. Then $du/dx = 6$ and $dx = du/6$. Substituting,

$$\int (6x-11)^{-5}\,dx = \int u^{-5}\frac{du}{6} = \frac{1}{6}\int u^{-5}\,du$$

Integrating, $\quad \dfrac{1}{6}\displaystyle\int u^{-5}\,du = \frac{1}{6}\left(\frac{1}{-4}u^{-4}\right) = -\frac{1}{24}u^{-4} + c$

Substituting, $\quad \displaystyle\int (6x-11)^{-5}\,dx = -\tfrac{1}{24}(6x-11)^{-4} + c$

Notice that here $du/dx = 6 \neq 1$, and the power rule cannot be used directly.

14.11. Redo Problem 14.7, given

$$\int \frac{x^2}{(4x^3+7)^2}\,dx$$

$$\int \frac{x^2}{(4x^3+7)^2}\,dx = \int x^2(4x^3+7)^{-2}\,dx$$

Let $u = 4x^3 + 7$, $du/dx = 12x^2$, and $dx = du/12x^2$. Substituting,

$$\int x^2 u^{-2} \frac{du}{12x^2} = \frac{1}{12}\int u^{-2}\,du$$

Integrating,
$$\frac{1}{12}\int u^{-2}\,du = -\frac{1}{12}u^{-1} + c$$

Substituting,
$$\int \frac{x^2}{(4x^3+7)^2}\,dx = -\frac{1}{12(4x^3+7)} + c$$

14.12. Redo Problem 14.7, given

$$\int \frac{6x^2 + 4x + 10}{(x^3 + x^2 + 5x)^3}\,dx$$

Let $u = x^3 + x^2 + 5x$. Then $du/dx = 3x^2 + 2x + 5$ and $dx = du/(3x^2 + 2x + 5)$. Substituting,

$$\int (6x^2 + 4x + 10)u^{-3} \frac{du}{3x^2 + 2x + 5} = 2\int u^{-3}\,du$$

Integrating,
$$2\int u^{-3}\,du = -u^{-2} + c$$

Substituting,
$$\int \frac{6x^2 + 4x + 10}{(x^3 + x^2 + 5x)^3}\,dx = -\frac{1}{(x^3 + x^2 + 5x)^2} + c$$

14.13. Redo Problem 14.7, given

$$\int \frac{dx}{9x - 5}$$

$$\int \frac{dx}{9x-5} = \int (9x - 5)^{-1}\,dx$$

Let $u = 9x - 5$, $du/dx = 9$, and $dx = du/9$. Substituting,

$$\int u^{-1}\frac{du}{9} = \frac{1}{9}\int u^{-1}\,du$$

Integrating with Rule 4, $\frac{1}{9}\int u^{-1}\,du = \frac{1}{9}\ln|u| + c$. Since u may be $\gtreqless 0$, and only positive numbers have logs, always use the absolute value of u. See Rule 4. Substituting,

$$\int \frac{dx}{9x-5} = \frac{1}{9}\ln|9x - 5| + c$$

14.14. Redo Problem 14.7, given

$$\int \frac{3x^2 + 2}{4x^3 + 8x}\,dx$$

Let $u = 4x^3 + 8x$, $du/dx = 12x^2 + 8$, and $dx = du/(12x^2 + 8)$. Substituting,

$$\int (3x^2 + 2)u^{-1} \frac{du}{12x^2 + 8} = \frac{1}{4}\int u^{-1}\,du$$

Integrating,
$$\frac{1}{4}\int u^{-1}\,du = \frac{1}{4}\ln|u| + c$$

Substituting, $$\int \frac{3x^2+2}{4x^3+8x}\,dx = \frac{1}{4}\ln|4x^3+8x| + c$$

14.15. Use the substitution method to find the integral for $\int x^3 e^{x^4}\,dx$. Check your answer.

Let $u = x^4$. Then $du/dx = 4x^3$ and $dx = du/4x^3$. Substituting, and noting that u is now an exponent,

$$\int x^3 e^u \frac{du}{4x^3} = \frac{1}{4}\int e^u\,du$$

Integrating with Rule 6, $$\frac{1}{4}\int e^u\,du = \frac{1}{4}e^u + c$$

Substituting, $$\int x^3 e^{x^4}\,dx = \frac{1}{4}e^{x^4} + c$$

14.16. Redo Problem 14.15, given $\int 24xe^{3x^2}\,dx$.

Let $u = 3x^2$, $du/dx = 6x$, and $dx = du/6x$. Substituting,

$$\int 24xe^u \frac{du}{6x} = 4\int e^u\,du$$

Integrating, $$4\int e^u\,du = 4e^u + c$$

Substituting, $$\int 24xe^{3x^2}\,dx = 4e^{3x^2} + c$$

14.17. Redo Problem 14.15, given $\int 14e^{2x+7}\,dx$.

Let $u = 2x+7$; then $du/dx = 2$ and $dx = du/2$. Substituting,

$$\int 14e^u \frac{du}{2} = 7\int e^u\,du = 7e^u + c$$

Substituting, $$\int 14e^{2x+7}\,dx = 7e^{2x+7} + c$$

14.18. Redo Problem 14.15, given $\int 5xe^{5x^2+3}\,dx$.

Let $u = 5x^2+3$, $du/dx = 10x$, and $dx = du/10x$. Substituting,

$$\int 5xe^u \frac{du}{10x} = \frac{1}{2}\int e^u\,du$$

Integrating, $$\frac{1}{2}\int e^u\,du = \frac{1}{2}e^u + c$$

Substituting, $$\int 5xe^{5x^2+3}\,dx = \frac{1}{2}e^{5x^2+3} + c$$

INTEGRATION BY PARTS

14.19. Use integration by parts to evaluate the following integral. Keep in the habit of checking your answers. Given $\int 15x(x+4)^{3/2}\,dx$.

Let $f(x) = 15x$, then $f'(x) = 15$. Let $g'(x) = (x+4)^{3/2}$, then $g(x) = \int (x+4)^{3/2}\,dx = \frac{2}{5}(x+4)^{5/2}$. Substituting in (*14.1*),

$$\int 15x(x+4)^{3/2}\,dx = f(x)g(x) - \int g(x)f'(x)\,dx$$

$$= 15x\left[\tfrac{2}{5}(x+4)^{5/2}\right] - \int \tfrac{2}{5}(x+4)^{5/2}15\,dx = 6x(x+4)^{5/2} - 6\int (x+4)^{5/2}\,dx$$

Evaluating the remaining integral,

$$\int 15x(x+4)^{3/2}\,dx = 6x(x+4)^{5/2} - \tfrac{12}{7}(x+4)^{7/2} + c$$

14.20. Redo Problem 14.19, given

$$\int \frac{2x}{(x-8)^3}\,dx$$

Let $f(x) = 2x$, $f'(x) = 2$, and $g'(x) = (x-8)^{-3}$; then $g(x) = \int (x-8)^{-3}\,dx = -\frac{1}{2}(x-8)^{-2}$. Substituting in (*14.1*),

$$\int \frac{2x}{(x-8)^3}\,dx = 2x\left[-\frac{1}{2}(x-8)^{-2}\right] - \int -\frac{1}{2}(x-8)^{-2}\quad 2\,dx = -x(x-8)^{-2} + \int (x-8)^{-2}\,dx$$

Integrating for the last time,

$$\int \frac{2x}{(x-8)^3}\,dx = -x(x-8)^{-2} - (x-8)^{-1} + c = \frac{-x}{(x-8)^2} - \frac{1}{x-8} + c$$

14.21. Redo Problem 14.19, given

$$\int \frac{5x}{(x-1)^2}\,dx$$

Let $f(x) = 5x$, $f'(x) = 5$, and $g'(x) = (x-1)^{-2}$; then $g(x) = \int (x-1)^{-2}\,dx = -(x-1)^{-1}$. Substituting in (*14.1*),

$$\int \frac{5x}{(x-1)^2}\,dx = 5x[-(x-1)^{-1}] - \int -(x-1)^{-1}5\,dx = -5x(x-1)^{-1} + 5\int (x-1)^{-1}\,dx$$

Integrating again,

$$\int \frac{5x}{(x-1)^2}\,dx = -5x(x-1)^{-1} + 5\ln|x-1| + c = \frac{-5x}{x-1} + 5\ln|x-1| + c$$

14.22. Redo Problem 14.19, given $\int 6xe^{x+7}\,dx$.

Let $f(x) = 6x$, $f'(x) = 6$, $g'(x) = e^{x+7}$, and $g(x) = \int e^{x+7}\,dx = e^{x+7}$. Using (*14.1*),

$$\int 6xe^{x+7}\,dx = 6xe^{x+7} - \int e^{x+7}6\,dx = 6xe^{x+7} - 6\int e^{x+7}\,dx$$

Integrating again,

$$\int 6xe^{x+7}\,dx = 6xe^{x+7} - 6e^{x+7} + c$$

14.23. Use integration by parts to evaluate $\int 16xe^{-(x+9)}\,dx$.

Let $f(x) = 16x$, $f'(x) = 16$, $g'(x) = e^{-(x+9)}$, and $g(x) = \int e^{-(x+9)}\,dx = -e^{-(x+9)}$. Using (*14.1*),

$$\int 16xe^{-(x+9)}\,dx = -16xe^{-(x+9)} - \int -e^{-(x+9)}16\,dx = -16xe^{-(x+9)} + 16\int e^{-(x+9)}\,dx$$

Integrating once more,

$$\int 16xe^{-(x+9)}\,dx = -16xe^{-(x+9)} - 16e^{-(x+9)} + c$$

14.24. Redo Problem 14.23, given $\int x^2 e^{2x}\,dx$.

Let $f(x) = x^2$, $f'(x) = 2x$, $g'(x) = e^{2x}$, and $g(x) = \int e^{2x}\,dx = \frac{1}{2}e^{2x}$. Substituting in (*14.1*).

$$\int x^2 e^{2x}\,dx = x^2(\tfrac{1}{2}e^{2x}) - \int \tfrac{1}{2}e^{2x}(2x)\,dx = \tfrac{1}{2}x^2 e^{2x} - \int xe^{2x}\,dx \qquad (14.2)$$

Using parts again for the remaining integral, $f(x) = x$, $f'(x) = 1$, $g'(x) = e^{2x}$, and $g(x) = \int e^{2x}\,dx = \frac{1}{2}e^{2x}$. Using (*14.1*),

$$\int xe^{2x}\,dx = x(\tfrac{1}{2}e^{2x}) - \int \tfrac{1}{2}e^{2x}\,dx = \tfrac{1}{2}xe^{2x} - \tfrac{1}{2}(\tfrac{1}{2}e^{2x})$$

Finally, substituting in (*14.2*),

$$\int x^2 e^{2x}\,dx = \tfrac{1}{2}x^2 e^{2x} - \tfrac{1}{2}xe^{2x} + \tfrac{1}{4}e^{2x} + c$$

ECONOMIC APPLICATIONS

14.25. The rate of net investment is $I = 40t^{3/5}$, and capital stock at $t = 0$ is 75. Find the capital function K.

$$K = \int I\,dt = \int 40t^{3/5}\,dt = 40(\tfrac{5}{8}t^{8/5}) + c = 25t^{8/5} + c$$

Substituting $t = 0$ and $K = 75$,

$$75 = 0 + c \qquad c = 75$$

Thus, $K = 25t^{8/5} + 75$.

14.26. The rate of net investment is $I = 60t^{1/3}$, and capital stock at $t = 1$ is 85. Find K.

$$K = \int 60t^{1/3}\,dt = 45t^{4/3} + c$$

At $t = 1$ and $K = 85$,

$$85 = 45(1) + c \qquad c = 40$$

Thus, $K = 45t^{4/3} + 40$.

14.27. Marginal cost is given by $\text{MC} = d\text{TC}/dQ = 25 + 30Q - 9Q^2$. Fixed cost is 55. Find the (*a*) total cost, (*b*) average cost, and (*c*) variable cost functions.

a)
$$\text{TC} = \int \text{MC}\,dQ = \int (25 + 30Q - 9Q^2)\,dQ = 25Q + 15Q^2 - 3Q^3 + c$$

With FC = 55, at $Q = 0$, TC = FC = 55. Thus, $c = \text{FC} = 55$ and $\text{TC} = 25Q + 15Q^2 - 3Q^3 + 55$.

b) $$\text{AC} = \frac{\text{TC}}{Q} = 25 + 15Q - 3Q^2 + \frac{55}{Q}$$

c) $$\text{VC} = \text{TC} - \text{FC} = 25Q + 15Q^2 - 3Q^3$$

14.28. Given $\text{MC} = d\text{TC}/dQ = 32 + 18Q - 12Q^2$, $\text{FC} = 43$. Find the (*a*) TC, (*b*) AC, and (*c*) VC functions.

a) $$\text{TC} = \int \text{MC}\,dQ = \int (32 + 18Q - 12Q^2)\,dQ = 32Q + 9Q^2 - 4Q^3 + c$$

At $Q = 0$, $\text{TC} = \text{FC} = 43$, $\text{TC} = 32Q + 9Q^2 - 4Q^3 + 43$.

b) $$\text{AC} = \frac{\text{TC}}{Q} = 32 + 9Q - 4Q^2 + \frac{43}{Q}$$

c) $$\text{VC} = \text{TC} - \text{FC} = 32Q + 9Q^2 - 4Q^3$$

14.29. Marginal revenue is given by $\text{MR} = d\text{TR}/dQ = 60 - 2Q - 2Q^2$. Find (*a*) the TR function and (*b*) the demand function $P = f(Q)$.

a) $$\text{TR} = \int \text{MR}\,dQ = \int (60 - 2Q - 2Q^2)\,dQ = 60Q - Q^2 - \tfrac{2}{3}Q^3 + c$$

At $Q = 0$, $\text{TR} = 0$. Therefore $c = 0$. Thus, $\text{TR} = 60Q - Q^2 - \frac{2}{3}Q^3$.

b) $\text{TR} = PQ$. Therefore, $P = \text{TR}/Q$, which is the same as saying that the demand function and the average revenue function are identical. Thus, $P = \text{AR} = \text{TR}/Q = 60 - Q - \frac{2}{3}Q^2$.

14.30. Find (*a*) the total revenue function and (*b*) the demand function, given

$$\text{MR} = 84 - 4Q - Q^2$$

a) $$\text{TR} = \int \text{MR}\,dQ = \int (84 - 4Q - Q^2)\,dQ = 84Q - 2Q^2 - \tfrac{1}{3}Q^3 + c$$

At $Q = 0$, $\text{TR} = 0$. Therefore $c = 0$. Thus, $\text{TR} = 84Q - 2Q^2 - \frac{1}{3}Q^3$.

b) $$P = \text{AR} = \frac{\text{TR}}{Q} = 84 - 2Q - \frac{1}{3}Q^2$$

14.31. With $C = f(Y)$, the marginal propensity to consume is given by $\text{MPC} = dC/dY = f'(Y)$. If the $\text{MPC} = 0.8$ and consumption is 40 when income is zero, find the consumption function.

$$C = \int f'(Y)\,dY = \int 0.8\,dY = 0.8Y + c$$

At $Y = 0$, $C = 40$. Thus, $c = 40$ and $C = 0.8Y + 40$.

14.32. Given $dC/dY = 0.6 + 0.1/\sqrt[3]{Y} = \text{MPC}$ and $C = 45$ when $Y = 0$. Find the consumption function.

$$C = \int \left(0.6 + \frac{0.1}{\sqrt[3]{Y}}\right) dY = \int (0.6 + 0.1Y^{-1/3})\,dY = 0.6Y + 0.15Y^{2/3} + c$$

At $Y = 0$, $C = 45$. Thus, $C = 0.6Y + 0.15Y^{2/3} + 45$.

14.33. The marginal propensity to save is given by $dS/dY = 0.5 - 0.2Y^{-1/2}$. There is dissaving of 3.5 when income is 25, that is, $S = -3.5$ when $Y = 25$. Find the savings function.

$$S = \int (0.5 - 0.2Y^{-1/2})\, dY = 0.5Y - 0.4Y^{1/2} + c$$

At $Y = 25$, $S = -3.5$.

$$-3.5 = 0.5(25) - 0.4(\sqrt{25}) + c \qquad c = -14$$

Thus, $S = 0.5Y - 0.4Y^{1/2} - 14$.

14.34. Given $\text{MC} = d\text{TC}/dQ = 12e^{0.5Q}$ and $\text{FC} = 36$. Find the total cost.

$$\text{TC} = \int 12e^{0.5Q}\, dQ = 12\frac{1}{0.5}e^{0.5Q} + c = 24e^{0.5Q} + c$$

With $\text{FC} = 36$, $\text{TC} = 36$ when $Q = 0$. Substituting, $36 = 24e^{0.5(0)} + c$. Since $e^0 = 1$, $36 = 24 + c$, and $c = 12$. Thus, $\text{TC} = 24e^{0.5Q} + 12$. Notice that c does not always equal FC.

14.35. Given $\text{MC} = 16e^{0.4Q}$ and $\text{FC} = 100$. Find TC.

$$\text{TC} = \int 16e^{0.4Q}\, dQ = 16\left(\frac{1}{0.4}\right)e^{0.4Q} + c = 40e^{0.4Q} + c$$

At $Q = 0$, $\text{TC} = 100$.

$$100 = 40e^0 + c \qquad c = 60$$

Thus, $\text{TC} = 40e^{0.4Q} + 60$.

CHAPTER 15

Integral Calculus: The Definite Integral

15.1 AREA UNDER A CURVE

There is no geometric formula for the area under an irregularly shaped curve, such as $y = f(x)$ between $x = a$ and $x = b$ in Fig. 15-1(a). If the interval $[a, b]$ is divided into n subintervals $[x_1, x_2]$, $[x_2, x_3]$, etc., and rectangles are constructed such that the height of each is equal to the smallest value of the function in the subinterval, as in Fig. 15-1(b), then the sum of the areas of the rectangles $\Sigma_{i=1}^{n} f(x_i)\,\Delta x_i$, called a *Riemann sum*, will approximate, but underestimate, the actual area under the curve. The smaller the subintervals (the smaller the Δx_i), the more rectangles are created and the closer the combined area of the rectangles $\Sigma_{i=1}^{n} f(x_i)\,\Delta x_i$ approaches the actual area under the curve. If

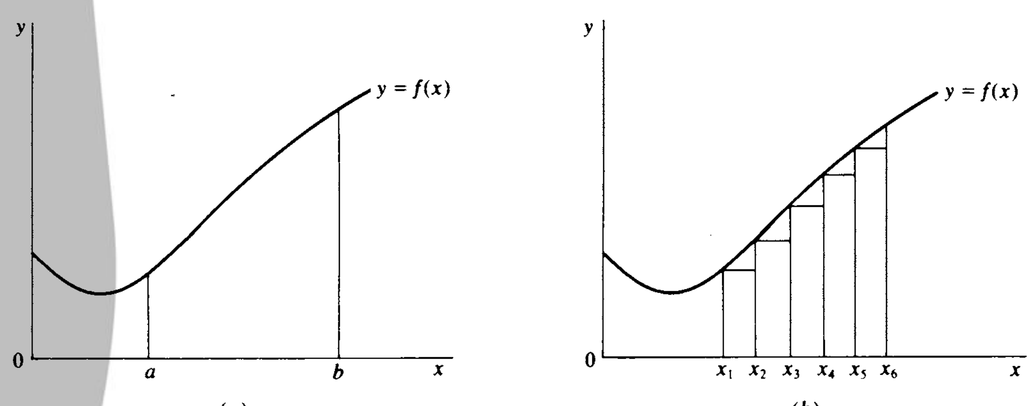

Fig. 15-1

the number of subintervals is increased so that $n \to \infty$, each subinterval becomes infinitesimal ($\Delta x_i = dx_i = dx$) and the area A under the curve can be expressed mathematically as

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x_i$$

15.2 THE DEFINITE INTEGRAL

The area under a graph of a continuous function such as that in Fig. 15-1 from a to b $(a < b)$ can be expressed more succinctly as the *definite integral* of $f(x)$ over the interval a to b. Put mathematically,

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x_i$$

Here the left-hand side is read, "the integral from a to b of f of $x\,dx$." Here a is called the *lower limit* of integration and b the *upper limit* of integration. Unlike the indefinite integral which is a set of functions containing all the antiderivatives of $f(x)$, as explained in Example 3 of Chapter 14, the definite integral is a real number which can be evaluated by using the fundamental theorem of calculus (Section 15.3).

15.3 THE FUNDAMENTAL THEOREM OF CALCULUS

The *fundamental theorem of calculus* states that the numerical value of the definite integral of a continuous function $f(x)$ over the interval from a to b is given by the indefinite integral $F(x) + c$ evaluated at the upper limit of integration b, minus the same indefinite integral $F(x) + c$ evaluated at the lower limit of integration a. Since c is common to both, the constant of integration is eliminated in subtraction. Expressed mathematically,

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$$

where the symbol $\big|_a^b$, $\big]_a^b$, or $[\cdots]_a^b$ indicates that b and a are to be substituted successively for x. See Examples 1 and 2 and Problems 15.1 to 15.10.

EXAMPLE 1. The definite integrals given below

$$(1)\quad \int_1^4 10x\,dx \qquad (2)\quad \int_1^3 (4x^3 + 6x)\,dx$$

are evaluated as follows:

1) $$\int_1^4 10x\,dx = 5x^2\Big|_1^4 = 5(4)^2 - 5(1)^2 = 75$$

2) $$\int_1^3 (4x^3 + 6x)\,dx = [x^4 + 3x^2]_1^3 = [(3)^4 + 3(3)^2] - [(1)^4 + 3(1)^2] = 108 - 4 = 104$$

EXAMPLE 2. The definite integral is used below to determine the area under the curve in Fig. 15-2 over the interval 0 to 20 as follows:

$$A = \int_0^{20} \tfrac{1}{2}x\,dx = \tfrac{1}{4}x^2\Big|_0^{20} = \tfrac{1}{4}(20)^2 - \tfrac{1}{4}(0)^2 = 100$$

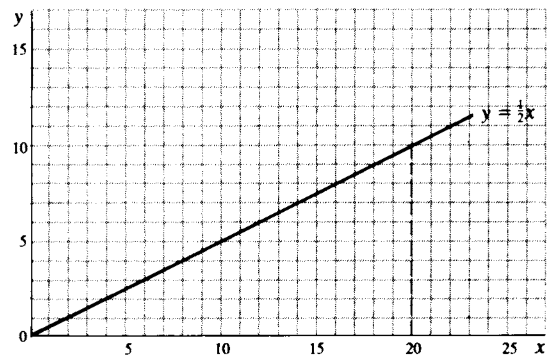

Fig. 15-2

The answer can be checked by using the geometric formula $A = \frac{1}{2}xy$:

$$A = \tfrac{1}{2}xy = \tfrac{1}{2}(20)(10) = 100$$

15.4 PROPERTIES OF DEFINITE INTEGRALS

1. Reversing the order of the limits changes the sign of the definite integral.

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx \tag{15.1}$$

2. If the upper limit of integration equals the lower limit of integration, the value of the definite integral is zero.

$$\int_a^a f(x)\,dx = F(a) - F(a) = 0 \tag{15.2}$$

3. The definite integral can be expressed as the sum of component subintegrals.

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx \qquad a \le b \le c \tag{15.3}$$

4. The sum or difference of two definite integrals with identical limits of integration is equal to the definite integral of the sum or difference of the two functions.

$$\int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx = \int_a^b [f(x) \pm g(x)]\,dx \tag{15.4}$$

5. The definite integral of a constant times a function is equal to the constant times the definite integral of the function.

$$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx \tag{15.5}$$

See Example 3 and Problems 15.11 to 15.14.

EXAMPLE 3. To illustrate a sampling of the properties presented above, the following definite integrals are evaluated:

1. $\int_1^3 2x^3\,dx = -\int_3^1 2x^3\,dx$

$$\int_1^3 2x^3\,dx = \tfrac{1}{2}x^4\Big|_1^3 = \tfrac{1}{2}(3)^4 - \tfrac{1}{2}(1)^4 = 40$$

   Checking this answer,

$$\int_3^1 2x^3\,dx = \tfrac{1}{2}x^4\Big|_3^1 = \tfrac{1}{2}(1)^4 - \tfrac{1}{2}(3)^4 = -40$$

2. $\int_5^5 (2x+3)\,dx = 0$

   Checking this answer,

$$\int_5^5 (2x+3)\,dx = [x^2+3x]_5^5 = [(5)^2+3(5)] - [(5)^2+3(5)] = 0$$

3. $\int_0^4 6x\,dx = \int_0^3 6x\,dx + \int_3^4 6x\,dx$

$$\int_0^4 6x\,dx = 3x^2\Big|_0^4 = 3(4)^2 - 3(0)^2 = 48$$

$$\int_0^3 6x\,dx = 3x^2\Big|_0^3 = 3(3)^2 - 3(0)^2 = 27$$

$$\int_3^4 6x\,dx = 3x^2\Big|_3^4 = 3(4)^2 - 3(3)^2 = 21$$

   Checking this answer, $48 = 27 + 21$

15.5 AREA BETWEEN CURVES

The area of a region between two or more curves can be evaluated by applying the properties of definite integrals outlined above. The procedure is demonstrated in Example 4 and treated in Problems 15.15 to 15.18.

EXAMPLE 4. Using the properties of integrals, the area of the region between two functions such as $y_1 = 3x^2 - 6x + 8$ and $y_2 = -2x^2 + 4x + 1$ from $x = 0$ to $x = 2$ is found in the following way:

a) Draw a rough sketch of the graph of the functions and shade in the desired area as in Fig. 15-3.

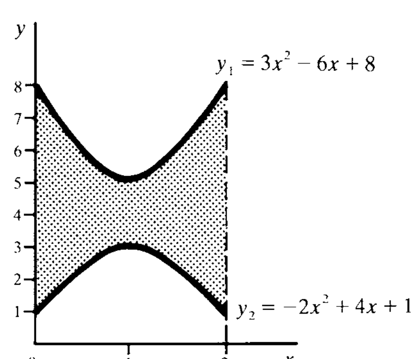

Fig. 15-3

b) Note the relationship between the curves. Since y_1 lies above y_2, the desired region is simply the area under y_1 minus the area under y_2 between $x = 0$ and $x = 2$. Hence,

$$A = \int_0^2 (3x^2 - 6x + 8)\,dx - \int_0^2 (-2x^2 + 4x + 1)\,dx$$

From (*15.4*),

$$A = \int_0^2 [(3x^2 - 6x + 8) - (-2x^2 + 4x + 1)]\,dx$$

$$= \int_0^2 (5x^2 - 10x + 7)\,dx$$

$$= \left(\tfrac{5}{3}x^3 - 5x^2 + 7x\right)\Big|_0^2 = 7\tfrac{1}{3} - 0 = 7\tfrac{1}{3}$$

15.6 IMPROPER INTEGRALS

The area under some curves that extend infinitely far along the x axis, as in Fig. 15-4(a), may be estimated with the help of improper integrals. A definite integral with infinity for either an upper or lower limit of integration is called an *improper integral*.

$$\int_a^\infty f(x)\,dx \qquad \text{and} \qquad \int_{-\infty}^b f(x)\,dx$$

are improper integrals because ∞ is not a number and cannot be substituted for x in $F(x)$. They can, however, be defined as the limits of other integrals, as shown below.

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx \qquad \text{and} \qquad \int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx$$

If the limit in either case exists, the improper integral is said to *converge*. The integral has a definite value, and the area under the curve can be evaluated. If the limit does not exist, the improper integral *diverges* and is meaningless. See Example 5 and Problems 15.19 to 15.25.

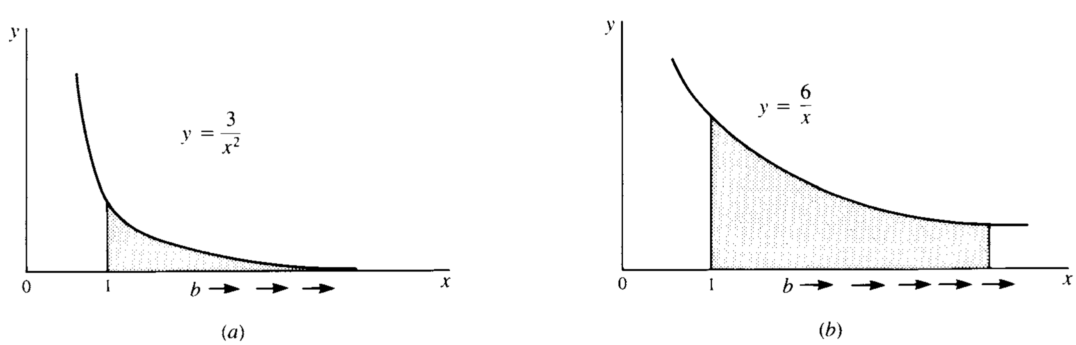

Fig. 15-4

EXAMPLE 5. The improper integrals given below

$$(a)\quad \int_1^\infty \frac{3}{x^2}\,dx \qquad (b)\quad \int_1^\infty \frac{6}{x}\,dx$$

are sketched in Fig. 15-4(a) and (b) and evaluated as follows:

a)

$$\int_1^\infty \frac{3}{x^2}\,dx = \lim_{b\to\infty}\int_1^b \frac{3}{x^2}\,dx = \lim_{b\to\infty}\left[\frac{-3}{x}\right]_1^b$$

$$= \lim_{b\to\infty}\left[\frac{-3}{b} - \frac{(-3)}{1}\right] = \lim_{b\to\infty}\left(\frac{-3}{b} + 3\right) = 3$$

because as $b \to \infty$, $-3/b \to 0$. Hence the improper integral is convergent and the area under the curve in Fig. 15-4(*a*) equals 3.

b)
$$\int_1^{\infty} \frac{6}{x}\,dx = \lim_{b\to\infty} \int_1^{b} \frac{6}{x}\,dx$$
$$= \lim_{b\to\infty} [6 \ln|x|]_1^b = \lim_{b\to\infty} [6 \ln|b| - 6 \ln|1|]$$
$$= \lim_{b\to\infty} [6 \ln|b|] \qquad \text{since} \qquad \ln|1| = 0$$

As $b \to \infty$, $6 \ln|b| \to \infty$. The improper integral diverges and has no definite value. The area under the curve in Fig. 15-4(*b*) cannot be computed even though the graph is deceptively similar to the one in (*a*).

15.7 L'HÔPITAL'S RULE

If the limit of a function $f(x) = g(x)/h(x)$ as $x \to a$ cannot be evaluated, such as (1) when both numerator and denominator approach zero, giving rise to the indeterminate form 0/0, or (2) when both numerator and denominator approach infinity, giving rise to the indeterminate form ∞/∞, *L'Hôpital's rule* can often be helpful. L'Hôpital's rule states:

$$\lim_{x\to a} \frac{g(x)}{h(x)} = \lim_{x\to a} \frac{g'(x)}{h'(x)} \tag{15.6}$$

It is illustrated in Example 6 and Problem 15.26.

EXAMPLE 6. The limits of the functions given below are found as follows, using L'Hôpital's rule. Note that numerator and denominator are differentiated separately, not as a quotient.

$$(a) \quad \lim_{x\to 4} \frac{x-4}{16-x^2} \qquad\qquad (b) \quad \lim_{x\to\infty} \frac{6x-2}{7x+4}$$

a) As $x \to 4$, $x - 4$ and $16 - x^2 \to 0$. Using (*15.6*), therefore, and differentiating numerator and denominator separately,

$$\lim_{x\to 4} \frac{x-4}{16-x^2} = \lim_{x\to 4} \frac{1}{-2x} = -\frac{1}{8}$$

b) As $x \to \infty$, both $6x - 2$ and $7x + 4 \to \infty$. Using (*15.6*),

$$\lim_{x\to\infty} \frac{6x-2}{7x+4} = \lim_{x\to\infty} \frac{6}{7} = \frac{6}{7}$$

15.8 CONSUMERS' AND PRODUCERS' SURPLUS

A demand function $P_1 = f_1(Q)$, as in Fig. 15-5(*a*), represents the different prices consumers are willing to pay for different quantities of a good. If equilibrium in the market is at (Q_0, P_0), then the consumers who would be willing to pay more than P_0 benefit. Total benefit to consumers is represented by the shaded area and is called *consumers' surplus*. Mathematically,

$$\text{Consumers' surplus} = \int_0^{Q_0} f_1(Q)\,dQ - Q_0 P_0 \tag{15.7}$$

A supply function $P_2 = f_2(Q)$, as in Fig. 15-5(*b*), represents the prices at which different quantities of a good will be supplied. If market equilibrium occurs at (Q_0, P_0), the producers who would supply at a lower price than P_0 benefit. Total gain to producers is called *producers' surplus* and is designated by the shaded area. Mathematically,

$$\text{Producers' surplus} = Q_0 P_0 - \int_0^{Q_0} f_2(Q)\,dQ \tag{15.8}$$

See Example 7 and Problems 15.27 to 15.31.

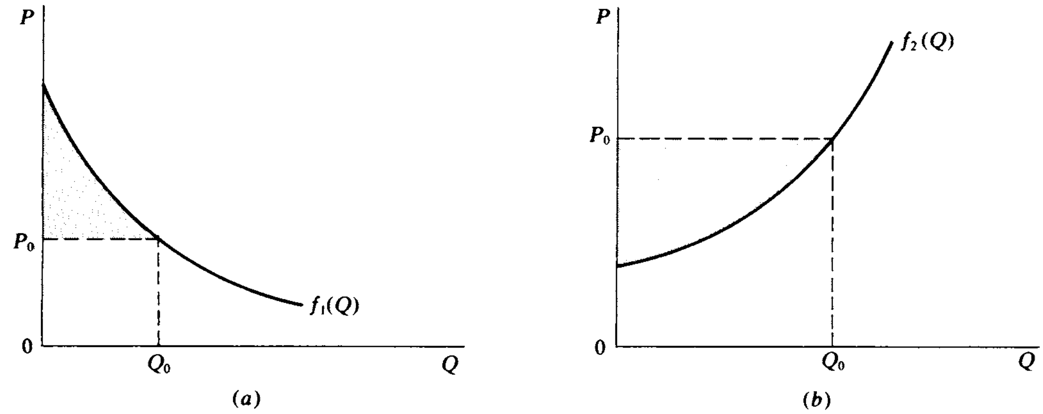

Fig. 15-5

EXAMPLE 7. Given the demand function $P = 42 - 5Q - Q^2$. Assuming that the equilibrium price is 6, the consumers' surplus is evaluated as follows:

At $P_0 = 6$,

$$42 - 5Q - Q^2 = 6$$
$$36 - 5Q - Q^2 = 0$$
$$(Q + 9)(-Q + 4) = 0$$

So $Q_0 = 4$, because $Q = -9$ is not feasible. Substituting in (*15.7*),

$$\begin{aligned}\text{Consumers' surplus} &= \int_0^4 (42 - 5Q - Q^2)\,dQ - (4)(6) \\ &= [42Q - 2.5Q^2 - \tfrac{1}{3}Q^3]_0^4 - 24 \\ &= (168 - 40 - 21\tfrac{1}{3}) - 0 - 24 = 82\tfrac{2}{3}\end{aligned}$$

15.9 THE DEFINITE INTEGRAL AND PROBABILITY

The probability P that an event will occur can be measured by the corresponding area under a probability density function. A *probability density* or *frequency function* is a continuous function $f(x)$ such that:

1. $f(x) \geq 0$. Probability cannot be negative.
2. $\int_{-\infty}^{\infty} f(x)\,dx = 1$. The probability of the event occurring over the entire range of x is 1.
3. $P(a < x < b) = \int_a^b f(x)\,dx$. The probability of the value of x falling within the interval $[a, b]$ is the value of the definite integral from a to b.

See Example 8 and Problems 15.32 and 15.33.

EXAMPLE 8. The time in minutes between cars passing on a highway is given by the frequency function $f(t) = 2e^{-2t}$ for $t \geq 0$. The probability of a car passing in 0.25 minute is calculated as follows:

$$P = \int_0^{0.25} 2e^{-2t}\,dt = -e^{-2t}\Big|_0^{0.25} = -e^{-0.5} - (-e^0) = -0.606531 + 1 = 0.393469$$

Solved Problems

DEFINITE INTEGRALS

15.1. Evaluate the following definite integrals:

a) $\int_0^6 5x\,dx$

$$\int_0^6 5x\,dx = 2.5x^2\Big|_0^6 = 2.5(6)^2 - 2.5(0)^2 = 90$$

b) $\int_1^{10} 3x^2\,dx$

$$\int_1^{10} 3x^2\,dx = x^3\Big|_1^{10} = (10)^3 - (1)^3 = 999$$

c) $\int_1^{64} x^{-2/3}\,dx$

$$\int_1^{64} x^{-2/3}\,dx = 3x^{1/3}\Big|_1^{64} = 3\sqrt[3]{64} - 3\sqrt[3]{1} = 9$$

d) $\int_1^3 (x^3 + x + 6)\,dx$

$$\int_1^3 (x^3 + x + 6)\,dx = \left(\tfrac{1}{4}x^4 + \tfrac{1}{2}x^2 + 6x\right)\Big|_1^3 = \tfrac{1}{4}(3)^4 + \tfrac{1}{2}(3)^2 + 6(3) - [\tfrac{1}{4}(1)^4 + \tfrac{1}{2}(1)^2 + 6(1)] = 36$$

e) $\int_1^4 (x^{-1/2} + 3x^{1/2})\,dx$

$$\int_1^4 (x^{-1/2} + 3x^{1/2})\,dx = (2x^{1/2} + 2x^{3/2})\Big|_1^4 = 2\sqrt{4} + 2\sqrt{4^3} - (2\sqrt{1} + 2\sqrt{1^3}) = 16$$

f) $\int_0^3 4e^{2x}\,dx$

$$\int_0^3 4e^{2x}\,dx = 2e^{2x}\Big|_0^3 = 2(e^{2(3)} - e^{2(0)})$$
$$= 2(403.4 - 1) = 804.8$$

g) $\int_0^{10} 2e^{-2x}\,dx$

$$\int_0^{10} 2e^{-2x}\,dx = -e^{-2x}\Big|_0^{10} = -e^{-2(10)} - (-e^{-2(0)}) = -e^{-20} + e^0 = 1$$

SUBSTITUTION METHOD

15.2. Use the substitution method to integrate the following definite integral:

$$\int_0^3 8x(2x^2 + 3)\,dx$$

Let $u = 2x^2 + 3$. Then $du/dx = 4x$ and $dx = du/4x$. Ignore the limits of integration for the moment, and treat the integral as an indefinite integral. Substituting in the original integrand,

$$\int 8x(2x^2+3)\,dx = \int 8xu\frac{du}{4x} = 2\int u\,du$$

Integrating with respect to u,

$$2\int u\,du = 2\left(\frac{u^2}{2}\right) + c = u^2 + c \tag{15.9}$$

Finally, by substituting $u = 2x^2 + 3$ in (*15.9*) and recalling that c will drop out in the integration, the definite integral can be written in terms of x, incorporating the original limits:

$$\int_0^3 8x(2x^2+3)\,dx = (2x^2+3)^2\Big|_0^3 = [2(3)^2+3]^2 - [2(0)^2+3]^2 = 441 - 9 = 432$$

Because in the original substitution $u \neq x$ but $2x^2 + 3$, the limits of integration in terms of x will differ from the limits of integration in terms of u. The limits can be expressed in terms of u, if so desired. Since we have set $u = 2x^2 + 3$ and x ranges from 0 to 3, the limits in terms of u are $u = 2(3)^2 + 3 = 21$ and $u = 2(0)^2 + 3 = 3$. Using these limits with the integral expressed in terms of u, as in (*15.9*),

$$2\int_3^{21} u\,du = u^2\Big|_3^{21} = 441 - 9 = 432$$

15.3. Redo Problem 15.2, given $\int_1^2 x^2(x^3-5)^2\,dx$.

Let $u = x^3 - 5$, $du/dx = 3x^2$, and $dx = du/3x^2$. Substituting independently of the limits,

$$\int x^2(x^3-5)^2\,dx = \int x^2u^2\frac{du}{3x^2} = \frac{1}{3}\int u^2\,du$$

Integrating with respect to u and ignoring the constant,

$$\tfrac{1}{3}\int u^2\,du = \tfrac{1}{3}(\tfrac{1}{3}u^3) = \tfrac{1}{9}u^3$$

Substituting $u = x^3 - 5$ and incorporating the limits for x,

$$\int_1^2 x^2(x^3-5)^2\,dx = [\tfrac{1}{9}(x^3-5)^3]_1^2$$
$$= \tfrac{1}{9}[(2)^3-5]^3 - \tfrac{1}{9}[(1)^3-5]^3 = \tfrac{1}{9}(27) - \tfrac{1}{9}(-64) = 10.11$$

Since $u = x^3 - 5$ and the limits for x are $x = 1$ and $x = 2$, by substitution the limits for u are $u = (1)^3 - 5 = -4$ and $u = (2)^3 - 5 = 3$. Incorporating these limits for the integral with respect to u,

$$\tfrac{1}{3}\int_{-4}^{3} u^2\,du = [\tfrac{1}{9}u^3]_{-4}^{3} = \tfrac{1}{9}(3)^3 - \tfrac{1}{9}(-4)^3 = 10.11$$

15.4. Redo Problem 15.2, given

$$\int_0^2 \frac{3x^2}{(x^3+1)^2}\,dx$$

Let $u = x^3 + 1$. Then $du/dx = 3x^2$ and $dx = du/3x^2$. Substituting,

$$\int \frac{3x^2}{(x^3+1)^2}\,dx = \int 3x^2u^{-2}\frac{du}{3x^2} = \int u^{-2}\,du$$

Integrating with respect to u and ignoring the constant,

$$\int u^{-2}\,du = -u^{-1}$$

Substituting $u = x^3 + 1$ with the original limits,

$$\int_0^2 \frac{3x^2}{(x^3+1)^2}\,dx = -(x^3+1)^{-1}\Big|_0^2 = \frac{-1}{2^3+1} - \frac{-1}{0^3+1} = -\frac{1}{9} + 1 = \frac{8}{9}$$

With $u = x^3 + 1$, and the limits of x ranging from 0 to 2, the limits of u are $u = (0)^3 + 1 = 1$ and $u = (2)^3 + 1 = 9$. Thus,

$$\int_1^9 u^{-2}\,du = -u^{-1}\Big|_1^9 = (-\tfrac{1}{9}) - (-\tfrac{1}{1}) = \tfrac{8}{9}$$

15.5. Integrate the following definite integral by means of the substitution method:

$$\int_0^3 \frac{6x}{x^2+1}\,dx$$

Let $u = x^2 + 1$, $du/dx = 2x$, and $dx = du/2x$. Substituting,

$$\int \frac{6x}{x^2+1}\,dx = \int 6xu^{-1}\frac{du}{2x} = 3\int u^{-1}\,du$$

Integrating with respect to u,

$$3\int u^{-1}\,du = 3\ln|u|$$

Substituting $u = x^2 + 1$,

$$\int_0^3 \frac{6x}{x^2+1}\,dx = 3\ln|x^2+1|\,\Big|_0^3$$

$$= 3\ln|3^2+1| - 3\ln|0^2+1| = 3\ln 10 - 3\ln 1$$

Since $\ln 1 = 0$, $\quad = 3\ln 10 = 6.9078$

The limits of u are $u = (0)^2 + 1 = 1$ and $u = (3)^2 + 1 = 10$. Integrating with respect to u,

$$3\int_1^{10} u^{-1}\,du = 3\ln|u|\,\Big|_1^{10} = 3\ln 10 - 3\ln 1 = 3\ln 10 = 6.9078$$

15.6. Redo Problem 15.5, given $\int_1^2 4xe^{x^2+2}\,dx$.

Let $u = x^2 + 2$. Then $du/dx = 2x$ and $dx = du/2x$. Substituting,

$$\int 4xe^{x^2+2}\,dx = \int 4xe^u\frac{du}{2x} = 2\int e^u\,du$$

Integrating with respect to u and ignoring the constant,

$$2\int e^u\,du = 2e^u$$

Substituting $u = x^2 + 2$,

$$\int_1^2 4xe^{x^2+2}\,dx = 2e^{x^2+2}\Big|_1^2 = 2(e^{(2)^2+2} - e^{(1)^2+2}) = 2(e^6 - e^3)$$

$$= 2(403.43 - 20.09) = 766.68$$

With $u = x^2 + 2$, the limits of u are $u = (1)^2 + 2 = 3$ and $u = (2)^2 + 2 = 6$.

$$2\int_3^6 e^u\,du = 2e^u\Big|_3^6 = 2(e^6 - e^3) = 766.68$$

15.7. Redo Problem 15.5, given $\int_0^1 3x^2 e^{2x^3+1}\,dx$.

Let $u = 2x^3 + 1$, $du/dx = 6x^2$, and $dx = du/6x^2$. Substituting,

$$\int 3x^2 e^{2x^3+1}\,dx = \int 3x^2 e^u \frac{du}{6x^2} = \frac{1}{2}\int e^u\,du$$

Integrating with respect to u,

$$\tfrac{1}{2}\int e^u\,du = \tfrac{1}{2}e^u$$

Substituting $u = 2x^3 + 1$,

$$\int_0^1 3x^2 e^{2x^3+1}\,dx = \tfrac{1}{2}e^{2x^3+1}\Big|_0^1 = \tfrac{1}{2}(e^3 - e^1) = \tfrac{1}{2}(20.086 - 2.718) = 8.684$$

With $u = 2x^3 + 1$, the limits of u are $u = 2(0)^3 + 1 = 1$ and $u = 2(1)^3 + 1 = 3$. Thus

$$\tfrac{1}{2}\int_1^3 e^u\,du = \tfrac{1}{2}e^u\Big|_1^3 = \tfrac{1}{2}(e^3 - e^1) = 8.68$$

INTEGRATION BY PARTS

15.8. Integrate the following definite integral, using the method of integration by parts:

$$\int_2^5 \frac{3x}{(x+1)^2}\,dx$$

Let $f(x) = 3x$; then $f'(x) = 3$. Let $g'(x) = (x+1)^{-2}$; then $g(x) = \int (x+1)^{-2}\,dx = -(x+1)^{-1}$. Substituting in (*14.1*),

$$\begin{aligned}\int \frac{3x}{(x+1)^2}\,dx &= 3x[-(x+1)^{-1}] - \int -(x+1)^{-1}3\,dx\\ &= -3x(x+1)^{-1} + 3\int (x+1)^{-1}\,dx\end{aligned}$$

Integrating and ignoring the constant,

$$\int \frac{3x}{(x+1)^2}\,dx = -3x(x+1)^{-1} + 3\ln|x+1|$$

Applying the limits,

$$\begin{aligned}\int_2^5 \frac{3x}{(x+1)^2}\,dx &= [-3x(x+1)^{-1} + 3\ln|x+1|]_2^5\\ &= \left[-\frac{3(5)}{5+1} + 3\ln|5+1|\right] - \left[-\frac{3(2)}{2+1} + 3\ln|2+1|\right]\\ &= -\tfrac{5}{2} + 3\ln 6 + 2 - 3\ln 3\\ &= 3(\ln 6 - \ln 3) - \tfrac{1}{2} = 3(1.7918 - 1.0986) - 0.5 = 1.5796\end{aligned}$$

15.9. Redo Problem 15.8, given

$$\int_1^3 \frac{4x}{(x+2)^3}\,dx$$

Let $f(x) = 4x$, $f'(x) = 4$, $g'(x) = (x+2)^{-3}$, and $g(x) = \int (x+2)^{-3}\,dx = -\frac{1}{2}(x+2)^{-2}$. Substituting in (*14.1*),

$$\int \frac{4x}{(x+2)^3}\,dx = 4x\left[-\frac{1}{2}(x+2)^{-2}\right] - \int -\frac{1}{2}(x+2)^{-2}4\,dx$$

$$= -2x(x+2)^{-2} + 2\int (x+2)^{-2}\,dx$$

Integrating,

$$\int \frac{4x}{(x+2)^3}\,dx = -2x(x+2)^{-2} - 2(x+2)^{-1}$$

Applying the limits,

$$\int_1^3 \frac{4x}{(x+2)^3}\,dx = [-2x(x+2)^{-2} - 2(x+2)^{-1}]_1^3$$

$$= [-2(3)(3+2)^{-2} - 2(3+2)^{-1}] - [-2(1)(1+2)^{-2} - 2(1+2)^{-1}]$$

$$= -\tfrac{6}{25} - \tfrac{2}{5} + \tfrac{2}{9} + \tfrac{2}{3} = \tfrac{56}{225}$$

15.10. Redo Problem 15.8, given $\int_1^3 5xe^{x+2}\,dx$.

Let $f(x) = 5x$, $f'(x) = 5$, $g'(x) = e^{x+2}$, and $g(x) = \int e^{x+2}\,dx = e^{x+2}$. Applying (*14.1*),

$$\int 5xe^{x+2}\,dx = 5xe^{x+2} - \int e^{x+2}5\,dx = 5xe^{x+2} - 5\int e^{x+2}\,dx$$

Integrating,

$$\int 5xe^{x+2}\,dx = 5xe^{x+2} - 5e^{x+2}$$

Applying the limits,

$$\int_1^3 5xe^{x+2}\,dx = [5xe^{x+2} - 5e^{x+2}]_1^3 = (15e^5 - 5e^5) - (5e^3 - 5e^3) = 10e^5 = 10(148.4) = 1484$$

PROPERTIES OF DEFINITE INTEGRALS

15.11. Show $\int_{-4}^4 (8x^3 + 9x^2)\,dx = \int_{-4}^0 (8x^3 + 9x^2)\,dx + \int_0^4 (8x^3 + 9x^2)\,dx$.

$$\int_{-4}^4 (8x^3 + 9x^2)\,dx = 2x^4 + 3x^3 \Big|_{-4}^{4} = 704 - 320 = 384$$

$$\int_{-4}^0 (8x^3 + 9x^2)\,dx = 2x^4 + 3x^3 \Big|_{-4}^{0} = 0 - 320 = -320$$

$$\int_0^4 (8x^3 + 9x^2)\,dx = 2x^4 + 3x^3 \Big|_{0}^{4} = 704 - 0 = 704$$

Checking this answer, $\quad -320 + 704 = 384$

15.12. Show $\int_0^{16}(x^{-1/2}+3x)\,dx = \int_0^4(x^{-1/2}+3x)\,dx + \int_4^9(x^{-1/2}+3x)\,dx + \int_9^{16}(x^{-1/2}+3x)\,dx$.

$$\int_0^{16}(x^{-1/2}+3x)\,dx = 2x^{1/2}+1.5x^2\Big|_0^{16} = 392-0 = 392$$

$$\int_0^{4}(x^{-1/2}+3x)\,dx = 2x^{1/2}+1.5x^2\Big|_0^{4} = 28-0 = 28$$

$$\int_4^{9}(x^{-1/2}+3x)\,dx = 2x^{1/2}+1.5x^2\Big|_4^{9} = 127.5-28 = 99.5$$

$$\int_9^{16}(x^{-1/2}+3x)\,dx = 2x^{1/2}+1.5x^2\Big|_9^{16} = 392-127.5 = 264.5$$

Checking this answer, $28+99.5+264.5=392$

15.13. Show

$$\int_0^3 \frac{6x}{x^2+1}\,dx = \int_0^1 \frac{6x}{x^2+1}\,dx + \int_1^2 \frac{6x}{x^2+1}\,dx + \int_2^3 \frac{6x}{x^2+1}\,dx$$

From Problem 15.5,

$$\int_0^3 \frac{6x}{x^2+1}\,dx = 3\ln|x^2+1|\Big|_0^3 = 3\ln 10$$

$$\int_0^1 \frac{6x}{x^2+1}\,dx = 3\ln|x^2+1|\Big|_0^1 = 3\ln 2 - 0 = 3\ln 2$$

$$\int_1^2 \frac{6x}{x^2+1}\,dx = 3\ln|x^2+1|\Big|_1^2 = 3\ln 5 - 3\ln 2$$

$$\int_2^3 \frac{6x}{x^2+1}\,dx = 3\ln|x^2+1|\Big|_2^3 = 3\ln 10 - 3\ln 5$$

Checking this answer, $3\ln 2 + 3\ln 5 - 3\ln 2 + 3\ln 10 - 3\ln 5 = 3\ln 10$

15.14. Show $\int_1^3 5xe^{x+2}\,dx = \int_1^2 5xe^{x+2}\,dx + \int_2^3 5xe^{x+2}\,dx$.

From Problem 15.10,

$$\int_1^3 5xe^{x+2}\,dx = [5xe^{x+2} - 5e^{x+2}]_1^3 = 10e^5$$

$$\int_1^2 5xe^{x+2}\,dx = [5xe^{x+2} - 5e^{x+2}]_1^2 = (10e^4-5e^4)-(5e^3-5e^3) = 5e^4$$

$$\int_2^3 5xe^{x+2}\,dx = [5xe^{x+2} - 5e^{x+2}]_2^3 = (15e^5-5e^5)-(10e^4-5e^4) = 10e^5-5e^4$$

Checking this answer, $5e^4+10e^5-5e^4 = 10e^5$

AREA BETWEEN CURVES

15.15. (*a*) Draw the graphs of the following functions, and (*b*) evaluate the area between the curves over the stated interval:

$$y_1 = 7-x \qquad \text{and} \qquad y_2 = 4x-x^2 \qquad \text{from } x=1 \text{ to } x=4$$

a) See Fig. 15-6.

b) From Fig. 15-6, the desired region is the area under the curve specified by $y_1 = 7-x$ from $x=1$ to

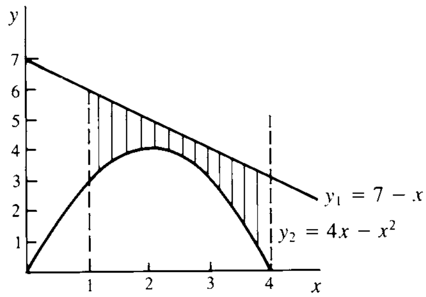

Fig. 15-6

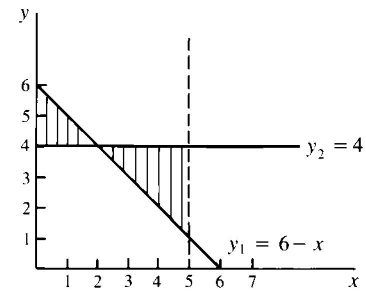

Fig. 15-7

$x = 4$ minus the area under the curve specified by $y_2 = 4x - x^2$ from $x = 1$ to $x = 4$. Using the properties of definite integrals,

$$A = \int_1^4 (7 - x)\,dx - \int_1^4 (4x - x^2)\,dx = \int_1^4 (x^2 - 5x + 7)\,dx$$
$$= [\tfrac{1}{3}x^3 - 2.5x^2 + 7x]_1^4$$
$$= [\tfrac{1}{3}(4)^3 - 2.5(4)^2 + 7(4)] - [\tfrac{1}{3}(1)^3 - 2.5(1)^2 + 7(1)] = 4.5$$

15.16. Redo Problem 15.15, given

$$y_1 = 6 - x \qquad \text{and} \qquad y_2 = 4 \qquad \text{from } x = 0 \text{ to } x = 5$$

Notice the shift in the relative positions of the curves at the point of intersection.

a) See Fig. 15-7.

b) From Fig. 15-7, the desired area is the area between $y_1 = 6 - x$ and $y_2 = 4$ from $x = 0$ to $x = 2$ plus the area between $y_2 = 4$ and $y_1 = 6 - x$ from $x = 2$ to $x = 5$. Mathematically,

$$A = \int_0^2 [(6 - x) - 4]\,dx + \int_2^5 [4 - (6 - x)]\,dx$$
$$= \int_0^2 (2 - x)\,dx + \int_2^5 (x - 2)\,dx$$
$$= [2x - \tfrac{1}{2}x^2]_0^2 + [\tfrac{1}{2}x^2 - 2x]_2^5 = 2 - 0 + 2.5 - (-2) = 6.5$$

15.17. Redo Problem 15.15, given

$$y_1 = x^2 - 4x + 8 \qquad \text{and} \qquad y_2 = 2x \qquad \text{from } x = 0 \text{ to } x = 3$$

a) See Fig. 15-8.

b)
$$A = \int_0^2 [(x^2 - 4x + 8) - 2x]\,dx + \int_2^3 [2x - (x^2 - 4x + 8)]\,dx$$
$$= \int_0^2 (x^2 - 6x + 8)\,dx + \int_2^3 (-x^2 + 6x - 8)\,dx$$
$$= [\tfrac{1}{3}x^3 - 3x^2 + 8x]_0^2 + [-\tfrac{1}{3}x^3 + 3x^2 - 8x]_2^3 = 7\tfrac{1}{3}$$

15.18. Redo Problem 15.15, given

$$y_1 = x^2 - 4x + 12 \qquad \text{and} \qquad y_2 = x^2 \qquad \text{from } x = 0 \text{ to } x = 4$$

a) See Fig. 15-9.

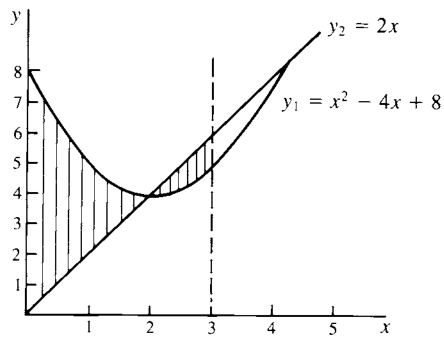

Fig. 15-8

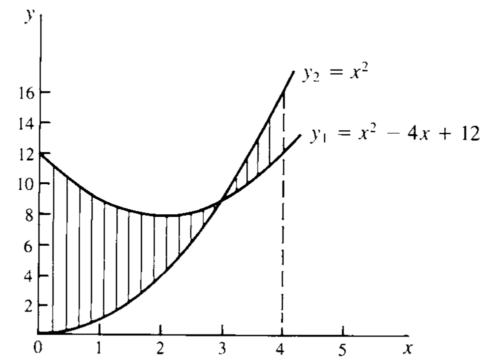

Fig. 15-9

b)
$$\begin{aligned} A &= \int_0^3 [(x^2-4x+12)-x^2]\,dx + \int_3^4 [x^2-(x^2-4x+12)]\,dx \\ &= \int_0^3 (12-4x)\,dx + \int_3^4 (4x-12)\,dx \\ &= [12x-2x^2]_0^3 + [2x^2-12x]_3^4 = 20 \end{aligned}$$

IMPROPER INTEGRALS AND L'HÔPITAL'S RULE

15.19. (*a*) Specify why the integral given below is improper and (*b*) test for convergence. Evaluate where possible.

$$\int_1^\infty \frac{2x}{(x^2+1)^2}\,dx$$

a) This is an example of an improper integral because the upper limit of integration is infinite.

b)
$$\int_1^\infty \frac{2x}{(x^2+1)^2}\,dx = \lim_{b\to\infty}\int_1^b \frac{2x}{(x^2+1)^2}\,dx$$

Let $u = x^2+1$, $du/dx = 2x$, and $dx = du/2x$. Substituting,

$$\int \frac{2x}{(x^2+1)^2}\,dx = \int 2xu^{-2}\frac{du}{2x} = \int u^{-2}\,du$$

Integrating with respect to u and ignoring the constant,

$$\int u^{-2}\,du = -u^{-1}$$

Substituting $u = x^2+1$ and incorporating the limits of x,

$$\begin{aligned} \int_1^\infty \frac{2x}{(x^2+1)^2}\,dx &= \lim_{b\to\infty}\int_1^b \frac{2x}{(x^2+1)^2}\,dx = -(x^2+1)^{-1}\Big|_1^b \\ &= \frac{-1}{b^2+1} + \frac{1}{(1)^2+1} = \frac{1}{2} - \frac{1}{b^2+1} \end{aligned}$$

As $b\to\infty$, $1/(b^2+1)\to 0$. The integral converges and has a value of $\frac{1}{2}$.

15.20. Redo Problem 15.19, given

$$\int_1^\infty \frac{dx}{x+7}$$

a) This is an improper integral because one of its limits of integration is infinite.

b)
$$\int_1^\infty \frac{dx}{x+7} = \lim_{b\to\infty}\int_1^b \frac{dx}{x+7} = \ln|x+7|\Big|_1^b$$
$$= \ln|b+7| - \ln|1+7|$$

As $b \to \infty$, $\ln|b+7| \to \infty$. The integral diverges and is meaningless.

15.21. Redo Problem 15.19, given $\int_{-\infty}^0 e^{3x}\,dx$.

a) The lower limit is infinite.

b)
$$\int_{-\infty}^0 e^{3x}\,dx = \lim_{a\to-\infty}\int_a^0 e^{3x}\,dx = \tfrac{1}{3}e^{3x}\Big|_a^0$$
$$= \tfrac{1}{3}e^{3(0)} - \tfrac{1}{3}e^{3a} = \tfrac{1}{3} - \tfrac{1}{3}e^{3a}$$

As $a \to -\infty$, $\frac{1}{3}e^{3a} \to 0$. The integral converges and has a value of $\frac{1}{3}$.

15.22. (*a*) Specify why the integral given below is improper and (*b*) test for convergence. Evaluate where possible:

$$\int_{-\infty}^0 (5-x)^{-2}\,dx$$

a) The lower limit is infinite.

b)
$$\int_{-\infty}^0 (5-x)^{-2}\,dx = \lim_{a\to-\infty}\int_a^0 (5-x)^{-2}\,dx$$

Let $u = 5 - x$, $du/dx = -1$, and $dx = -du$. Substituting,

$$\int (5-x)^{-2}\,dx = \int u^{-2}(-du) = -\int u^{-2}\,du$$

Integrating with respect to u,

$$-\int u^{-2}\,du = u^{-1}$$

Substituting $u = 5 - x$ and incorporating the limits of x,

$$\int_{-\infty}^0 (5-x)^{-2}\,dx = \lim_{a\to-\infty}\int_a^0 (5-x)^{-2}\,dx = (5-x)^{-1}\Big|_a^0$$
$$= \frac{1}{5-0} - \frac{1}{5-a} = \frac{1}{5} - \frac{1}{5-a}$$

As $a \to -\infty$, $1/(5-a) \to 0$. The integral converges and equals $\frac{1}{5}$.

15.23. Redo Problem 15.22, given $\int_{-\infty}^0 2xe^x\,dx$.

a) The lower limit is infinite.

b)
$$\int_{-\infty}^0 2xe^x\,dx = \lim_{a\to-\infty}\int_a^0 2xe^x\,dx$$

Using integration by parts, let $f(x) = 2x$, $f'(x) = 2$, $g'(x) = e^x$, and $g(x) = \int e^x\,dx = e^x$. Substituting in (*14.1*),

$$\int 2xe^x\,dx = 2xe^x - \int e^x 2\,dx$$

Integrating once again,

$$\int 2xe^x\,dx = 2xe^x - 2e^x$$

Incorporating the limits,

$$\begin{aligned}\int_{-\infty}^{0} 2xe^x\,dx &= \lim_{a\to-\infty}\int_a^0 2xe^x\,dx = (2xe^x - 2e^x)\Big|_a^0 \\ &= [2(0)e^0 - 2e^0] - (2ae^a - 2e^a) \\ &= -2 - 2ae^a + 2e^a \qquad \text{since} \quad e^0 = 1\end{aligned}$$

As $a \to -\infty$, $e^a \to 0$. Therefore the integral converges and has a value of -2.

15.24. Redo Problem 15.22, given

$$\int_0^6 \frac{dx}{x-6}$$

a) This is also an improper integral because, as x approaches 6 from the left ($x \to 6^-$), the integrand $\to -\infty$.

b)
$$\begin{aligned}\int_0^6 \frac{dx}{x-6} &= \lim_{b\to6}\int_0^b \frac{dx}{x-6} = \ln|x-6|\Big|_0^b \\ &= \ln|b-6| - \ln|0-6|\end{aligned}$$

As $b \to 6^-$, $|b-6| \to 0$ and $\ln 0$ is undefined. Therefore, the integral diverges and is meaningless.

15.25. Redo Problem 15.22, given $\int_0^8 (8-x)^{-1/2}\,dx$.

a) As $x \to 8^-$, the integrand approaches infinity.

b)
$$\begin{aligned}\int_0^8 (8-x)^{-1/2}\,dx &= \lim_{b\to8}\int_0^b (8-x)^{-1/2}\,dx = -2(8-x)^{1/2}\Big|_0^b \\ &= (-2\sqrt{8-b}) - (-2\sqrt{8-0}) = 2\sqrt{8} - 2\sqrt{8-b}\end{aligned}$$

As $b \to 8^-$, $-2\sqrt{8-b} \to 0$. The integral converges and has a value of $2\sqrt{8} = 4\sqrt{2}$.

15.26. Use L'Hôpital's rule to evaluate the following limits:

a) $\displaystyle\lim_{x\to\infty}\frac{5x-9}{e^x}$

As $x \to \infty$, both $5x - 9$ and e^x tend to ∞, giving rise to the indeterminate form ∞/∞. Using (*15.6*), therefore, and differentiating numerator and denominator separately,

$$\lim_{x\to\infty}\frac{5x-9}{e^x} = \lim_{x\to\infty}\frac{5}{e^x} = \frac{5}{\infty} = 0$$

b) $\displaystyle\lim_{x\to\infty}\frac{1-e^{1/x}}{1/x}$

As $x \to \infty$, $1 - e^{1/x}$ and $1/x \to 0$. Using (*15.6*), therefore, and recalling that $1/x = x^{-1}$,

$$\lim_{x\to\infty} \frac{1-e^{1/x}}{1/x} = \lim_{x\to\infty} \frac{-(-1/x^2)e^{1/x}}{-1/x^2}$$

Simplifying algebraically,

$$\lim_{x\to\infty} \frac{1-e^{1/x}}{1/x} = \lim_{x\to\infty} (-e^{1/x}) = -e^0 = -1$$

c) $\lim_{x\to\infty} \dfrac{\ln 2x}{e^{5x}}$

As $x \to \infty$, $\ln 2x$ and $e^{5x} \to \infty$. Again using (*15.6*),

$$\lim_{x\to\infty} \frac{\ln 2x}{e^{5x}} = \lim_{x\to\infty} \frac{1/x}{5e^{5x}} = \frac{0}{\infty} = 0 \qquad \text{since } \frac{0}{\infty} \text{ is not an indeterminate form.}$$

d) $\lim_{x\to\infty} \dfrac{6x^3-7}{3x^2+9}$

$$\lim_{x\to\infty} \frac{6x^3-7}{3x^2+9} = \lim_{x\to\infty} \frac{18x^2}{6x} = \lim_{x\to\infty} 3x = \infty$$

e) $\lim_{x\to\infty} \dfrac{3x^2-7x}{4x^2-21}$

$$\lim_{x\to\infty} \frac{3x^2-7x}{4x^2-21} = \lim_{x\to\infty} \frac{6x-7}{8x}$$

Whenever application of L'Hôpital's rule gives rise to a new quotient whose limit is also an indeterminate form, L'Hôpital's rule must be applied again. Thus,

$$\lim_{x\to\infty} \frac{6x-7}{8x} = \lim_{x\to\infty} \frac{6}{8} = \frac{3}{4} \qquad \text{See Problem 3.4}(c).$$

f) $\lim_{x\to\infty} \dfrac{8x^3-5x^2+13x}{2x^3+7x^2-18x}$

Using L'Hôpital's rule repeatedly,

$$\lim_{x\to\infty} \frac{8x^3-5x^2+13x}{2x^3+7x^2-18x} = \lim_{x\to\infty} \frac{24x^2-10x+13}{6x^2+14x-18} = \lim_{x\to\infty} \frac{48x-10}{12x+14}$$
$$= \lim_{x\to\infty} \tfrac{48}{12} = 4$$

CONSUMERS' AND PRODUCERS' SURPLUS

15.27. Given the demand function $P = 45 - 0.5Q$, find the consumers' surplus CS when $P_0 = 32.5$ and $Q_0 = 25$.

Using (*15.7*),

$$\text{CS} = \int_0^{25} (45 - 0.5Q)\, dQ - (32.5)(25) = [45Q - 0.25Q^2]_0^{25} - 812.5$$
$$= [45(25) - 0.25(25)^2] - 0 - 812.5 = 156.25$$

15.28. Given the supply function $P = (Q + 3)^2$, find the producers' surplus PS at $P_0 = 81$ and $Q_0 = 6$.

From (*15.8*),

$$\text{PS} = (81)(6) - \int_0^6 (Q+3)^2\,dQ = 486 - [\tfrac{1}{3}(Q+3)^3]_0^6$$
$$= 486 - [\tfrac{1}{3}(6-3)^3 - \tfrac{1}{3}(0+3)^3] = 252$$

15.29. Given the demand function $P_d = 25 - Q^2$ and the supply function $P_s = 2Q + 1$. Assuming pure competition, find (*a*) the consumers' surplus and (*b*) the producers' surplus.

For market equilibrium, $s = d$. Thus,

$$2Q + 1 = 25 - Q^2 \qquad Q^2 + 2Q - 24 = 0$$
$$(Q+6)(Q-4) = 0 \qquad Q_0 = 4 \qquad P_0 = 9$$

since Q_0 cannot equal -6.

a)
$$\text{CS} = \int_0^4 (25 - Q^2)\,dQ - (9)(4) = [25Q - \tfrac{1}{3}Q^3]_0^4 - 36$$
$$= [25(4) - \tfrac{1}{3}(4)^3] - 0 - 36 = 42.67$$

b)
$$\text{PS} = (9)(4) - \int_0^4 (2Q+1)\,dQ$$
$$= 36 - [Q^2 + Q]_0^4 = 16$$

15.30. Given the demand function $P_d = 113 - Q^2$ and the supply function $P_s = (Q + 1)^2$ under pure competition, find (*a*) CS and (*b*) PS.

Multiplying the supply function out and equating supply and demand,

$$Q^2 + 2Q + 1 = 113 - Q^2 \qquad 2(Q^2 + Q - 56) = 0$$
$$(Q+8)(Q-7) = 0 \qquad Q_0 = 7 \qquad P_0 = 64$$

a)
$$\text{CS} = \int_0^7 (113 - Q^2)\,dQ - (64)(7) = [113Q - \tfrac{1}{3}Q^3]_0^7 - 448 = 228.67$$

b)
$$\text{PS} = (64)(7) - \int_0^7 (Q+1)^2\,dQ = 448 - [\tfrac{1}{3}(Q+1)^3]_0^7 = 448 - (170.67 - 0.33) = 277.67$$

15.31. Under a monopoly, the quantity sold and market price are determined by the demand function. If the demand function for a profit-maximizing monopolist is $P = 274 - Q^2$ and $\text{MC} = 4 + 3Q$, find the consumers' surplus.

Given $P = 274 - Q^2$,

$$\text{TR} = PQ = (274 - Q^2)Q = 274Q - Q^3$$

and
$$\text{MR} = \frac{d\text{TR}}{dQ} = 274 - 3Q^2$$

The monopolist maximizes profit at MR = MC. Thus,

$$274 - 3Q^2 = 4 + 3Q \qquad 3(Q^2 + Q - 90) = 0$$
$$(Q+10)(Q-9) = 0 \qquad Q_0 = 9 \qquad P_0 = 193$$

and
$$\text{CS} = \int_0^9 (274 - Q^2)\,dQ - (193)(9) = [274Q - \tfrac{1}{3}Q^3]_0^9 - 1737 = 486$$

FREQUENCY FUNCTIONS AND PROBABILITY

15.32. The probability in minutes of being waited on in a large chain restaurant is given by the frequency function $f(t) = \frac{4}{81}t^3$ for $0 \le t \le 3$. What is the probability of being waited on between 1 and 2 minutes?

$$P = \int_1^2 \tfrac{4}{81}t^3\,dt = \tfrac{1}{81}t^4 \Big|_1^2 = \tfrac{1}{81}(16) - \tfrac{1}{81}(1) = 0.1852$$

15.33. The proportion of assignments completed within a given day is described by the probability density function $f(x) = 12(x^2 - x^3)$ for $0 \le x \le 1$. What is the probability that (*a*) 50 percent or less of the assignments will be completed within the day and (*b*) 50 percent or more will be completed?

a)
$$P_a = \int_0^{0.5} 12(x^2 - x^3)\,dx = 12\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^{0.5}$$
$$= 12\left[\left(\frac{0.125}{3} - \frac{0.0625}{4}\right) - 0\right] = 0.3125$$

b)
$$P_b = \int_{0.5}^{1} 12(x^2 - x^3)\,dx = 12\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_{0.5}^{1}$$
$$= 12\left[\left(\frac{1}{3} - \frac{1}{4}\right) - \left(\frac{0.125}{3} - \frac{0.0625}{4}\right)\right] = 0.6875$$

As expected, $P_a + P_b = 0.3125 + 0.6875 = 1$.

OTHER ECONOMIC APPLICATIONS

15.34. Given $I(t) = 9t^{1/2}$, find the level of capital formation in (*a*) 8 years and (*b*) for the fifth through the eighth years (interval [4, 8]).

a)
$$K = \int_0^8 9t^{1/2}\,dt = 6t^{3/2}\Big|_0^8 = 6(8)^{3/2} - 0 = 96\sqrt{2} = 135.76$$

b)
$$K = \int_4^8 9t^{1/2}\,dt = 6t^{3/2}\Big|_4^8 = 6(8)^{3/2} - 6(4)^{3/2} = 135.76 - 48 = 87.76$$